JOURNAL OF APPROXIMATION THEORY 74, 266–279 (1993)

A Convergence Result in Nodal Spline Interpolation

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Communicated by Günther Nürnberger

Received May 29, 1991; accepted in revised form April 15, 1992

In this paper we study the property of uniform convergence for a certain nodal spline interpolation operator. After solving for the corresponding B-spline coefficients from a transposed Vandermonde linear system, we derive a Jackson-type estimate in which the dependence of the associated constants on order and mesh ratio has been explicitly calculated. Sufficient conditions for uniform convergence are then easily obtained.

1. Introduction

In the construction of spline approximation operators if often seems desirable to obtain the three properties of locality, interpolation, and optimal polynomial reproduction. However, it was shown in [2, pp. 109–110] that, in the case where the knots of the spline space are chosen to coincide with the interpolation points, the two properties of locality and interpolation are incompatible for quadratic and higher order splines. Employing a procedure based on the introduction of additional knots, De Villiers and Rohwer [2, 3] constructed and demonstrated, for arbitrary order, a so-called *nodal* spline approximation operator which was shown to indeed possess the desired three properties. Similar approaches have been followed by authors like, for quadratic splines, Neumann and Schmidt [4, pp. 307–309], and for arbitrary order splines, Dahmen, Goodman, and Micchelli [1].

The principal aim of this paper is to establish a Jackson-type estimate, as well as resulting sufficient conditions for uniform convergence, for the nodal spline interpolant constructed in [2, 3]. In accordance with the usage in [3], we henceforth employ the terminology *primary* and *secondary* knots for, respectively, the above-mentioned interpolation-points and additional knots.

0021-9045/93 $5.00

2. Notation and Preliminaries

Suppose that $[a, b]$ is a given finite interval on the real line $\mathbb{R}$, and for a given integer $m \geqslant 2$, let the integer n satisfy $n \geqslant m-1$. Let $\mathbb{Z}$ denote the set of integers and suppose $\{x_i; i \in \mathbb{Z}\}$ is a strict increasing sequence of real numbers, but with the choices $x_0 = a$, $x_{(m-1)n} = b$. We define the partition Δ of $\mathbb{R}$ by

$$\Delta := \{x_i; i \in \mathbb{Z}\}.$$

Next, we introduce the notation

$$\xi_{m,i} := x_{(m-1)i}, \qquad i \in \mathbb{Z},$$

and define the corresponding partition Π_m of $\mathbb{R}$ by

$$\Pi_m := \{\xi_{m,i}; i \in \mathbb{Z}\},$$

so that clearly $\Pi_m \subset \Delta$, with $\xi_{m,0} = a$, $\xi_{m,n} = b$. We call the points of Π_m and $\Delta \setminus \Pi_m$, respectively, the *primary* and *secondary* knots corresponding to the partition Δ for a given value of m.

We write $\mathbb{P}^m$ for the set of polynomials of order m (degree $\leqslant m-1$), whereas the symbol $S_m(\Delta)$ denotes the set of polynomial splines of order m and with simple knots at the points of Δ, so that $s \in S_m(\Delta)$ if and only if

$$s(x) = p_i(x), \qquad x \in [x_i, x_{i+1}], \quad i \in \mathbb{Z},$$

with $p_i \in \mathbb{P}^m$, $i \in \mathbb{Z}$, and $s \in C^{m-2}(\mathbb{R})$. We write $B[a, b]$ for the set of real-valued functions defined on $[a, b]$ and employ the usual symbol δ_{ij} for the Kronecker delta.

In the paper [3] it was proved constructively that there exists a *local* spline approximation operator W: $B[a, b] \to S_m(\Delta)$ with the properties of *interpolation at the primary knots,*

$$(Wf)(\xi_{m,i}) = f(\xi_{m,i}), \qquad i = 0, 1, ..., n, \quad f \in B[a, b],$$

as well as *optimal polynomial reproduction,*

$$Wp = p, \qquad p \in \mathbb{P}^m. \tag{1}$$

Noting that, in the linear case $m = 2$, Wf is trivially given by the broken line interpolant of f, *we henceforth assume that* $m \geqslant 3$. Indeed, for these values of m, it was shown in [3] that the defining formula for $(Wf)(x)$ on $[a, b]$ is given by

$$(Wf)(x)=\sum_{i=p_j}^{q_j} f(\xi_{m,i})\, w_{m,i}(x), \qquad x\in[\xi_{m,j}, \xi_{m,j+1}],\ j=0, 1, ..., n-1, \tag{2}$$

where

$$p_j := \max\{0, j-i_1+1\}, \qquad q_j := \min\{n, j+i_0\}, \tag{3}$$

and with the integers i_0 and i_1 defined by

$$i_0 := \begin{cases} \frac{1}{2}(m+1), & m \text{ odd}, \\ \frac{1}{2}m+1, & m \text{ even}, \end{cases} \qquad i_1 := (m+1)-i_0. \tag{4}$$

In the representation (2), the relevant values on $[a, b]$ of the functions $w_{m,i}$ can be calculated from the formulas

$$w_{m,i}(x)=\begin{cases} \prod_{k=0,k\neq i}^{m-1} \dfrac{x-\xi_{m,k}}{\xi_{m,i}-\xi_{m,k}}, & x\in[a, \xi_{m,i_1-1}], \\ s_{m,i}(x), & x\in[\xi_{m,i_1-1}, \xi_{m,n-i_0+1}], \\ \prod_{k=0,k\neq n-i}^{m-1} \dfrac{x-\xi_{m,n-k}}{\xi_{m,i}-\xi_{m,n-k}}, & x\in[\xi_{m,n-i_0+1}, b], \end{cases} \tag{5}$$

where, according to (4), the middle line of (5) is only applicable if $n\geqslant m$. Moreover, in (5), $s_{m,i}$ belongs to the set of *nodal splines* $\{s_{m,i}; i\in\mathbb{Z}\}\subset S_m(\Delta)$, as constructed and studied in the paper [2], where each $s_{m,i}$ has compact support $[\xi_{m,i-i_0}, \xi_{m,i+i_1}]$ and is *nodal* with respect to Π_m in the sense that $s_{m,i}(\xi_{m,j})=\delta_{ij}$, $j\in\mathbb{Z}$. In addition, the B-spline series for $s_{m,i}$ has the form

$$s_{m,i}(x)=\sum_{r=0}^{m-2} \sum_{j=-i_0}^{-i_0+(m-1)} \alpha_{i,r,j} B^m_{(m-1)(i+j)+r}(x), \qquad x\in\mathbb{R},\ i\in\mathbb{Z}, \tag{6}$$

with $\{B^m_k; k\in\mathbb{Z}\}\subset S_m(\Delta)$ denoting the *normalized* B-*splines* as defined in [5, p. 241]. In the series (6), for fixed $i\in\mathbb{Z}$ and $r\in\{0, 1, ..., m-2\}$, the coefficient set $\{\alpha_{i,r,j}; j=-i_0, -i_0+1, ..., -i_0+(m-1)\}$ can be characterized as the solution of the (nonsingular) $m\times m$ linear system

$$\sum_{j=-i_0}^{-i_0+(m-1)} (\xi_{m,i-j})^k\, \alpha_{i-j,r,j}=\beta_{i,r,k}, \qquad k=0, 1, ..., m-1, \tag{7}$$

with

$$\beta_{i,r,k} := \frac{(-1)^k k!}{(m-1)!} \psi^{(m-1-k)}_{i,r}(0), \qquad \psi_{i,r}(x) := \prod_{t=1}^{m-1} [x-x_{(m-1)i+r+t}]. \tag{8}$$

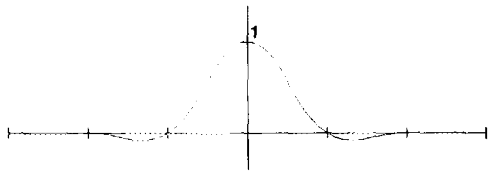

FIG. 1. The quadratic nodal spline $s_{3,i}$, as given by the B-spline series (6), in the case of an equidistant partition Δ.

As an illustrative example we have drawn, in Fig. 1, the quadratic nodal spline $s_{3,i}$ in the case where Δ is an equidistant partition.

Next, for the given primary partition $a=\xi_{m,0}<\xi_{m,1}<\cdots<\xi_{m,n}=b$ of $[a,b]$, we define the *primary norm* $H_{m,n}$ and *local primary mesh ratio* $R_{m,n}$ by

$$H_{m,n} := \max_{1\leqslant i\leqslant n} |\xi_{m,i}-\xi_{m,i-1}|, \tag{9}$$

$$R_{m,n} := \max_{1\leqslant i,j\leqslant n;|i-j|=1} \frac{|\xi_{m,i}-\xi_{m,i-1}|}{|\xi_{m,j}-\xi_{m,j-1}|}. \tag{10}$$

Note in particular that $R_{m,n}\geqslant 1$, with equality if and only if the primary partition is uniform.

Suppose $f\in C[a,b]$ is given, and define the associated error function $e_{m,n}$ by

$$e_{m,n}(x) := f(x)-(Wf)(x), \qquad x\in[a,b]. \tag{11}$$

Our purpose here is to prove that there exist positive numbers c_1 and c_2, both depending only on m and $R_{m,n}$, such that the maximum norm $\|e_{m,n}\|_\infty := \max_{a\leqslant x\leqslant b}|e_{m,n}(x)|$ satisfies the Jackson-type estimate

$$\|e_{m,n}\|_\infty \leqslant c_1\omega(f; c_2 H_{m,n}), \tag{12}$$

with the modulus of continuity ω defined by

$$\omega(f;\delta) := \max_{|x-y|\leqslant\delta;\, x,y\in[a,b]} |f(x)-f(y)|. \tag{13}$$

It is immediately clear from (12) that if, for a fixed value of m, we have a sequence (in n) of primary partitions for which $H_{m,n}\to 0$, $n\to\infty$, and also c_1 and c_2 are uniformly bounded in n, then the uniform convergence result $\|e_{m,n}\|_\infty\to 0$, $n\to\infty$, holds. Note in particular that the bound in (12) is *independent of the placement of the secondary knots.*

3. An Explicit Formula for the B-Spline Coefficients $\alpha_{i,r,j}$

Our first step toward the desired result (12) is to solve explicitly from the linear system (7) for the B-spline coefficients $\{\alpha_{i,r,j}\}$ appearing in (6). Indeed, recognizing the coefficient matrix corresponding to (7) as the transpose of a Vandermonde matrix, we deduce, in analogy to the situation in [2, pp. 113–115], the formula

$$\alpha_{i,r,j} = \frac{1}{(m-1)!} \frac{\sum_{k=0}^{m-1} (-1)^k \phi_{i+j,j}^{(k)}(0)\, \psi_{i+j,r}^{(m-1-k)}(0)}{\phi_{i+j,j}(\xi_{m,i})}, \tag{14}$$

with the polynomial $\phi_{i,j} \in \mathbb{P}^m$ defined by

$$\phi_{i,j}(x) := \prod_{v=-i_0;\, v \neq j}^{-i_0+(m-1)} (x - \xi_{m,i-v}). \tag{15}$$

It should be pointed out that, whereas in [2] the linear system (7) was first transformed into an equivalent form [2, Eq. (3.25)] before being solved, we find it more convenient here to solve (7) directly.

We show next that the formula (14) can be expressed entirely in terms of *differences between knots*. Indeed, defining the integer sets

$$\begin{aligned} M_j &:= \{-i_0, -i_0+1, ..., -i_0+(m-1)\} \backslash \{j\}, \\ j &= -i_0, -i_0+1, ..., -i_0+(m-1), \end{aligned} \tag{16}$$

we have the following result.

Theorem 3.1. *The B-spline coefficients in (6) are given by the expression*

$$\alpha_{i,r,j} = \frac{1}{(m-1)!} \times \sum_{1 \leqslant v_k \leqslant m \quad 1, k \in M_j;\, v_k \text{ distinct}} \prod_{k=-i_0;\, k \neq j}^{-i_0+(m-1)} \frac{\xi_{m,i+j-k} - x_{(m-1)(i+j)+r+v_k}}{\xi_{m,i+j-k} - \xi_{m,i}}. \tag{17}$$

The proof of Theorem 3.1 depends on the following two lemmas.

Lemma 3.2. *Let μ denote a positive integer and suppose $\{\gamma_1, \gamma_2, ..., \gamma_\mu\} \subset \mathbb{R}$. Then*

$$\prod_{r=1}^{\mu} (x - \gamma_r) = \sum_{k=0}^{\mu} (-1)^k \sigma_k(\gamma_1, \gamma_2, ..., \gamma_\mu)\, x^{\mu-k}, \tag{18}$$

with

$$\sigma_k(\gamma_1, \gamma_2, ..., \gamma_\mu) := \begin{cases} \sum_{i_1=1}^{\mu-k+1} \gamma_{i_1} \sum_{i_2=i_1+1}^{\mu-k+2} \gamma_{i_2} \cdots \sum_{i_k=i_{k-1}+1}^{\mu} \gamma_{i_k}, & k=1, 2, ..., \mu, \\ 1, & k=0. \end{cases} \tag{19}$$

Proof. Since (18) clearly holds for $\mu = 1$, it suffices to show that, if (18) holds with μ replaced by some given $\nu \in \{1, 2, ..., \mu - 1\}$, then it also holds with μ replaced by $\nu + 1$.

Now clearly, from the inductive hypothesis,

$$\begin{aligned} \prod_{r=1}^{\nu+1} (x-\gamma_r) &= \left[\sum_{k=0}^{\nu} (-1)^k \sigma_k(\gamma_1, ..., \gamma_\nu) x^{\nu-k} \right] [x - \gamma_{\nu+1}] \\ &= x^{\nu+1} + \sum_{k=1}^{\nu} (-1)^k [\sigma_k(\gamma_1, ..., \gamma_\nu) + \gamma_{\nu+1} \sigma_{k-1}(\gamma_1, ..., \gamma_\nu)] x^{\nu+1-k} \\ &\quad + (-1)^{\nu+1} \sigma_{\nu+1}(\gamma_1, ..., \gamma_{\nu+1}), \end{aligned}$$

by virtue of the definition (19), together with the fact that

$$\gamma_{\nu+1} \sigma_\nu(\gamma_1, ..., \gamma_\nu) = \prod_{k=1}^{\nu+1} \gamma_k = \sigma_{\nu+1}(\gamma_1, ..., \gamma_{\nu+1}).$$

Hence it remain to prove the identity

$$\sigma_k(\gamma_1, ..., \gamma_\nu) + \gamma_{\nu+1} \sigma_{k-1}(\gamma_1, ..., \gamma_\nu) = \sigma_k(\gamma_1, ..., \gamma_{\nu+1}), \qquad k = 1, 2, ..., \nu. \tag{20}$$

But, from (19),

$$\begin{aligned} \sigma_k(\gamma_1, ..., \gamma_{\nu+1}) &= \sum_{i_1=1}^{\nu+2-k} \gamma_{i_1} \sum_{i_2=i_1+1}^{\nu+3-k} \gamma_{i_2} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{\nu} \gamma_{i_{k-1}} \left[\sum_{i_k=i_{k-1}+1}^{\nu} \gamma_{i_k} + \gamma_{\nu+1} \right] \\ &= \sum_{i_1=1}^{\nu+1-k} \gamma_{i_1} \sum_{i_2=i_1+1}^{\nu+2-k} \gamma_{i_1} \cdots \sum_{i_k=i_{k-1}+1}^{\nu} \gamma_{i_k} \\ &\quad + \gamma_{\nu+1} \sum_{i_1=1}^{\nu-(k-1)+1} \gamma_{i_1} \sum_{i_2=i_1+1}^{\nu-(k-1)+2} \gamma_{i_1} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{\nu} \gamma_{i_{k-1}}, \end{aligned}$$

immediately yielding the desired result (20). Q.E.D.

Lemma 3.3. *Let μ denote a positive integer and suppose $\{\alpha_1, \alpha_2, ..., \alpha_\mu\}$ and $\{\beta_1, \beta_2, ..., \beta_\mu\}$ are subsets of $\mathbb{R}$. Then the polynomials $p, q \in \mathbb{P}^\mu$ defined by*

$$p(x) := \prod_{r=1}^{\mu} (x-\alpha_r), \qquad q(x) := \prod_{r=1}^{\mu} (x-\beta_r) \tag{21}$$

satisfy the identity

$$\begin{aligned} &\sum_{k=0}^{\mu} (-1)^k p^{(k)}(0)\, q^{(\mu-k)}(0) \\ &\quad = (-1)^\mu \sum_{i_1=1}^{\mu} (\alpha_{i_1}-\beta_1) \sum_{i_2=1; i_2\neq i_1}^{\mu} (\alpha_{i_2}-\beta_2) \\ &\qquad \cdots \sum_{i_\mu=1; i_\mu \neq i_1, ..., i_{\mu-1}}^{\mu} (\alpha_{i_\mu}-\beta_\mu). \end{aligned} \tag{22}$$

Proof. Applying the identity (18) to the polynomials in (21) we easily find that

$$\begin{aligned} &\sum_{k=0}^{\mu} (-1)^k p^{(k)}(0)\, q^{(\mu-k)}(0) \\ &\quad = (-1)^\mu \sum_{k=0}^{\mu} (-1)^k k!(\mu-k)!\, \sigma_{\mu-k}(\alpha_1, ..., \alpha_\mu)\, \sigma_k(\beta_1, ..., \beta_\mu). \end{aligned}$$

Hence, if we adopt the vector notation $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_\mu)$, and define the function $F: \mathbb{R}^\mu \to \mathbb{R}$ by

$$F(\boldsymbol{\beta}) := \sum_{i_1=1}^{\mu} (\alpha_{i_1}-\beta_1) \sum_{i_2=1: i_2\neq i_1}^{\mu} (\alpha_{i_2}-\beta_2) \cdots \sum_{i_\mu=1; i_\mu\neq i_1, ..., i_{\mu-1}}^{\mu} (\alpha_{i_\mu}-\beta_\mu), \tag{23}$$

it will clearly suffice to prove that

$$F(\boldsymbol{\beta}) = \sum_{k=0}^{\mu} (-1)^k k!(\mu-k)!\, \sigma_{\mu-k}(\alpha_1, ..., \alpha_\mu)\, \sigma_k(\beta_1, ..., \beta_\mu). \tag{24}$$

Our method of proof consists of showing that the right-hand side of (24) is precisely the multivariate Taylor expansion with respect to the origin ($\boldsymbol{\beta} = \mathbf{0}$) of F.

First, we see from (23) and (19) that

$$F(\mathbf{0}) = \mu!\, \sigma_\mu(\alpha_1, ..., \alpha_\mu). \tag{25}$$

Next, observing that the order in which the components of $\boldsymbol{\beta}$ appear in the definition (23) can be permutated arbitrarily without changing $F(\boldsymbol{\beta})$, we calculate the derivatives

$$\frac{\partial F}{\partial \beta_i}(\mathbf{0}) = -(\mu-1)!\,\sigma_{\mu-1}(\alpha_1, ..., \alpha_\mu), \qquad i=1,2,...,\mu,$$
$$\frac{\partial^2 F}{\partial \beta_j\,\partial \beta_i}(\mathbf{0}) = \begin{cases} 2!(\mu-2)!\,\sigma_{\mu-2}(\alpha_1, ..., \alpha_\mu), & j \neq i, \\ 0, & j = i. \end{cases} \tag{26}$$

In general, for $k \in \{1, 2, ..., \mu\}$, and with v_i, $i = 1, 2, ..., k$, denoting a sequence of integers such that $1 \leqslant v_i \leqslant \mu$, v_i distinct, we deduce from (23) and (19) that

$$\frac{\partial^k F}{\partial \beta_{v_1}\,\partial \beta_{v_2} \cdots \partial \beta_{v_k}}(\mathbf{0}) = (-1)^k\, k!(\mu-k)!\,\sigma_{\mu-k}(\alpha_1, ..., \alpha_\mu). \tag{27}$$

Finally, we combine the results (25), (26), and (27) to calculate the desired Taylor expansion; we find that

$$\begin{aligned} F(\boldsymbol{\beta}) &= \sum_{k=0}^{\mu} \frac{1}{k!} \sum_{1 \leqslant v_i \leqslant \mu, i=1,2,...,k; v_i \text{ distinct}} \frac{\partial^k F}{\partial \beta_{v_1}\,\partial \beta_{v_2} \cdots \partial \beta_{v_k}}(\mathbf{0})\, \beta_{v_1}\beta_{v_2}\cdots\beta_{v_k} \\ &= \sum_{k=0}^{\mu} \frac{1}{k!} [(-1)^k\, k!(\mu-k)!\,\sigma_{\mu-k}(\alpha_1, ..., \alpha_\mu)] \\ &\quad \times \sum_{1 \leqslant v_i \leqslant \mu, i=1,...,k; v_i \text{ distinct}} \beta_{v_1}\beta_{v_2}\cdots\beta_{v_k}, \end{aligned}$$

immediately yielding (24). Q.E.D.

Proof of Theorem 3.1. First note that one can rewrite the formula (14) in the form

$$\alpha_{i,r,j} = \frac{(-1)^{m-1}}{(m-1)!} \frac{\sum_{k=0}^{m-1} (-1)^k\, \psi_{i+j,r}^{(k)}(0)\, \phi_{i+j,j}^{(m-1-k)}(0)}{\prod_{k=-i_0,k\neq j}^{-i_0+(m-1)} (\xi_{m,i} - \xi_{m,i+j-k})},$$

having used also the definition (15). Now apply the identity (22) with the choices $\mu = m-1$, $p = \psi_{i+j,r}$, and $q = \phi_{i+j,j}$. Q.E.D.

4. A Uniform Bound for the Nodal Spline $s_{m,i}$

The specific representation formula (17) of the B-spline coefficients enables us to calculate a bound for $|\alpha_{i,r,j}|$ which depends only on the order m and the local primary mesh ratio $R_{m,n}$. The next (rather technical) result is used extensively.

LEMMA 4.1. *Let μ and ν be integers with $0 \leqslant \mu \leqslant n$, $0 \leqslant \nu \leqslant n$. Then*

$$|\xi_{m,\mu} - \xi_{m,\nu}| \leqslant \left[\sum_{\lambda=0}^{|\mu-\nu|-1} (R_{m,n})^{\lambda} \right] \begin{cases} |\xi_{m,\mu} - \xi_{m,\mu-1}|, & \mu > \nu, \\ |\xi_{m,\mu} - \xi_{m,\mu+1}|, & \mu < \nu. \end{cases} \tag{28}$$

Proof. Suppose first $\mu > \nu$, whence

$$|\xi_{m,\mu} - \xi_{m,\nu}| \leqslant \sum_{\lambda=0}^{\mu-\nu-1} |\xi_{m,\mu-\lambda} - \xi_{m,\mu-\lambda-1}|,$$

and the top line of (28) follows after repeated use of the fact that

$$|\xi_{m,i} - \xi_{m,i-1}| \leqslant R_{m,n} |\xi_{m,i+1} - \xi_{m,i}|, \qquad i = 1, 2, ..., n-1,$$

as can easily be seen from the definition (10). The case $\mu < \nu$ is proved similarly. Q.E.D.

THEOREM 4.2. *The* B-*spline coefficients in* (6) *are uniformly bounded by*

$$|\alpha_{i,r,j}| \leqslant \left[\sum_{\lambda=1}^{i_0} (R_{m,n})^{\lambda} \right]^{m-1}, \tag{29}$$

with the integer i_0 given as in (4).

Proof. Consider the representation formula (17) of $\alpha_{i,r,j}$. With the usual convention of writing $\lfloor c \rfloor$ for the largest integer less than or equal to a given real number c, we define the integer

$$\mu := \lfloor -(r + \nu_k)/(m-1) \rfloor, \tag{30}$$

for which, by virtue of the fact that $r \in \{0, 1, ..., m-2\}$ and $\nu_k \in \{1, 2, ..., m-1\}$ in (17), it is easily shown that $\mu \in \{-2, -1\}$. Thus, recalling the restriction $m \geqslant 3$, and noting from (4) that $i_0 \geqslant 2$ for $m \geqslant 3$, we deduce the inequalities

$$-i_0 \leqslant \mu, \qquad -i_0 + (m-1) \geqslant \mu + 1,$$

from which it then follows that the integer sets I and J defined by

$$I := \{-i_0, -i_0 + 1, ..., \mu\}, \qquad J := \{\mu + 1, \mu + 2, ..., -i_0 + (m-1)\}, \tag{31}$$

are indeed non-empty.

Noting also that, in (17),

$$1 \leqslant r + \nu_k \leqslant 2m - 3 < 2(m-1),$$

we then easily show that, for given i, r, and j in (17),

$$|\xi_{m,i+j-k}-x_{(m-1)(i+j)+r+v_k}|\leqslant\begin{cases}|\xi_{m,i+j-k}-\xi_{m,i+j}|, & k\in I,\\ |\xi_{m,i+j-k}-\xi_{m,i+j+2}|, & k\in J.\end{cases} \tag{32}$$

Now we apply the inequalities (28) to the right-hand side of (32) to obtain the bounds

$$\begin{aligned}&|\xi_{m,i+j-k}-x_{(m-1)(i+j)+r+v_k}|\\ &\quad\leqslant\begin{cases}\left[\sum_{\lambda=0}^{i_0-1}(R_{m,n})^{\lambda}\right]|\xi_{m,i+j-k}-\xi_{m,i+j-k-1}|, & k\in I,\\ \left[\sum_{\lambda=0}^{m-i_0}(R_{m,n})^{\lambda}\right]|\xi_{m,i+j-k}-\xi_{m,i+j-k+1}|, & k\in J,\end{cases}\end{aligned} \tag{33}$$

after having used also the bounds on k implied by the definitions in (31) of I and J. Also, in (17), it is clear that, for given i and j,

$$\begin{aligned}&|\xi_{m,i+j-k}-\xi_{m,i}|\\ &\quad\geqslant\begin{cases}|\xi_{m,i+j-k}-\xi_{m,i+j-k-1}|\\ \quad\geqslant(R_{m,n})^{-1}|\xi_{m,i+j-k}-\xi_{m,i+j-k+1}|, & -i_0\leqslant k<j,\\ |\xi_{m,i+j-k}-\xi_{m,i+j-k+1}|\\ \quad\geqslant(R_{m,n})^{-1}|\xi_{m,i+j-k}-\xi_{m,i+j-k-1}|, & j<k\leqslant -i_0+(m-1),\end{cases}\end{aligned} \tag{34}$$

where we have also used the definition (10) of $R_{m,n}$. Noting from (4) that

$$m-i_0\leqslant i_0-1 \qquad \text{for} \quad m\geqslant 3,$$

we see that the desired result (29) follows by combining (17), (33), and (34). Q.E.D.

It is now an easy matter to prove that the nodal spline $s_{m,i}$, as it appears in the middle line of (5), is uniformly bounded in the following sense.

THEOREM 4.3. *The nodal spline* $s_{m,i}$, *as defined by the* B-*spline series* (6), *is uniformly bounded by*

$$|s_{m,i}(x)|\leqslant\left[\sum_{\lambda=1}^{i_0}(R_{m,n})^{\lambda}\right]^{m-1}, \qquad a\leqslant x\leqslant b,\ i=0,1,...,n, \tag{35}$$

with the integer i_0 *given as in* (4).

Proof. From (6) and (29) we have, for $x \in [a, b]$ and $i \in \{0, 1, ..., n\}$,

$$|s_{m,i}(x)| \leqslant \left[\sum_{\lambda=1}^{i_0} (R_{m,n})^{\lambda} \right]^{m-1} \sum_{r=0}^{m-2} \sum_{j=-i_0}^{-i_0+(m-1)} B^m_{(m-1)(i+j)+r}(x), \tag{36}$$

having exploited also the known property (see [5, p. 243]) $B^m_k(x) \geqslant 0$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$, of B-splines. But, recalling the definition in (4) of the integer i_1, we see that

$$\begin{aligned} &\sum_{r=0}^{m-2} \sum_{j=-i_0}^{-i_0+(m-1)} B^m_{(m-1)(i+j)+r}(x) \\ &\quad = \sum_{j=-(m-1)i_0}^{(m-1)i_1-m} B^m_{(m-1)i+j}(x) \leqslant \sum_{k \in \mathbb{Z}} B^m_k(x) = 1, \end{aligned} \tag{37}$$

since $\{B^m_k(x); k \in \mathbb{Z}\}$ form a partition of unity for all $x \in \mathbb{R}$ (see [5, p. 243]), and the result (35) follows from (36) aınd (37). Q.E.D.

5. A Jackson-Type Estimate

In order to obtain a Jackson-type estimate of the form (12) for the error $e_{m,n} := f - Wf$, we next employ the result of Theorem 4.3 to establish an analogous uniform bound for the modified nodal spline $w_{m,i}$ appearing in the definition (2) of $(Wf)(x)$.

THEOREM 5.1. *The modified nodal spline $w_{m,i}$, as defined on $[a, b]$ by the formulas (5), is uniformly bounded by*

$$|w_{m,i}(x)| \leqslant \left[\sum_{\lambda=1}^{m-1} (R_{m,n})^{\lambda} \right]^{m-1}, \qquad a \leqslant x \leqslant b, \quad i = 0, 1, ..., n. \tag{38}$$

Proof. We observe from the definition (4) of i_0 that

$$i_0 \leqslant m - 1 \qquad \text{for} \quad m \geqslant 3,$$

and thus it follows from Theorem 4.3 that it will suffice to prove the inequality

$$\begin{aligned} &\prod_{k=0;k \neq i}^{m-1} \frac{|x - \xi_{m,k}|}{|\xi_{m,i} - \xi_{m,k}|} \\ &\quad \leqslant \left[\sum_{\lambda=1}^{m-1} (R_{m,n})^{\lambda} \right]^{m-1}, \qquad a \leqslant x \leqslant \xi_{m,i_1-1}, \quad i = 0, 1, ..., n, \end{aligned} \tag{39}$$

and similarly for the third line of (5).

Suppose $x \in [a, \xi_{m,i_1-1}]$. Then, in the left-hand side of (39), and for fixed k, we get

$$|x-\xi_{m,k}| \leqslant \sum_{\lambda=0}^{m-2} (R_{m,n})^{\lambda} \begin{cases} |\xi_{m,k}-\xi_{m,k-1}|, & x<\xi_{m,k}, \\ |\xi_{m,k}-\xi_{m,k+1}|, & x>\xi_{m,k}, \end{cases} \tag{40}$$

by virtue of the estimates (28), whereas, in analogy to (34), also

$$|\xi_{m,i}-\xi_{m,k}| \geqslant \begin{cases} |\xi_{m,k}-\xi_{m,k+1}| \geqslant (R_{m,n})^{-1} |\xi_{m,k}-\xi_{m,k-1}|, & i<k, \\ |\xi_{m,k}-\xi_{m,k-1}| \geqslant (R_{m,n})^{-1} |\xi_{m,k}-\xi_{m,k+1}|, & i>k. \end{cases} \tag{41}$$

The desired bound (39) then follows from (40) and (41). The third line of (5) is treated similarly. Q.E.D.

The following error estimate then holds.

THEOREM 5.2. *Suppose $f \in C[a,b]$. The corresponding error function $e_{m,n} := f - Wf$ then satisfies the Jackson-type estimate*

$$\|e_{m,n}\|_{\infty} \leqslant (m+1) \left[\sum_{\lambda=1}^{m-1} (R_{m,n})^{\lambda} \right]^{m-1} \omega(f; mH_{m,n}). \tag{42}$$

Proof. Fix the index $j \in \{0, 1, ..., n-1\}$, and let $x \in [\xi_{m,j}, \xi_{m,j+1}]$. Noting first from (1) and (2) that

$$\sum_{i=p_j}^{q_j} w_{m,i}(x) = 1,$$

we deduce that

$$|e_{m,n}(x)| = \left| \sum_{i=p_j}^{q_j} [f(x)-f(\xi_{m,i})]\, w_{m,i}(x) \right|$$

$$\leqslant \max_{p_j \leqslant i \leqslant q_j} |f(x)-f(\xi_{m,i})| \sum_{i=p_j}^{q_j} |w_{m,i}(x)|. \tag{43}$$

Here, from (38),

$$\sum_{i=p_j}^{q_j} |w_{m,i}(x)| \leqslant (q_j-p_j+1) \left[\sum_{\lambda=1}^{m-1} (R_{m,n})^{\lambda} \right]^{m-1}. \tag{44}$$

Next, using (3) and (4) it is easily shown that

$$q_j - p_j \leqslant m, \qquad j = 0, 1, ..., n-1. \tag{45}$$

Also, in (43),

$$\max_{p_j \leqslant i \leqslant q_j} |f(x) - f(\xi_{m,i})| \leqslant \omega(f; \max_{p_j \leqslant i \leqslant q_j} |x - \xi_{m,i}|), \tag{46}$$

by virtue of the definition (13) of ω. Using (3) and (4), we see next that $[\xi_{m,j}, \xi_{m,j+1}] \subset [\xi_{m,p_j}, \xi_{m,q_j}]$, and thus

$$\begin{aligned}\max_{p_j \leqslant i \leqslant q_j} |x - \xi_{m,i}| &\leqslant |\xi_{m,p_j} - \xi_{m,q_j}| \\ &\leqslant \sum_{k=p_j+1}^{q_j} |\xi_{m,k} - \xi_{m,k-1}| \\ &\leqslant (q_j - p_j)\, H_{m,n} \leqslant mH_{m,n},\end{aligned} \tag{47}$$

having noted the definition (9) of $H_{m,n}$, as well as the estimate (45).

Now combine the results (43) to (47), obtaining the estimate

$$\max_{\xi_{m,j} \leqslant x \leqslant \xi_{m,j+1}} |e_{m,n}(x)| \leqslant (m+1) \left[\sum_{\lambda=1}^{m-1} (R_{m,n})^{\lambda} \right]^{m-1} \omega(f; mH_{m,n}), \tag{48}$$

and the desired inequality (42) follows by virtue of the fact that the right-hand side of (48) is independent of j. Q.E.D.

Recalling the fundamental modulus of continuity property

$$\omega(f; \delta) \to 0, \delta \to 0 \qquad \text{if} \quad f \in C[a, b],$$

we can now immediately deduce, from (42), the following sufficient conditions for uniform convergence.

COROLLARY 5.3. *Suppose $f \in C[a, b]$, and, for fixed m, let*

$$a = \xi_{m,0} < \xi_{m,1} < \cdots < \xi_{m,n} = b, \qquad n = m-1, m, ...,$$

be a sequence of primary partitions such that

(a) $H_{m,n} \to 0$, $n \to \infty$;

(b) $R_{m,n} \leqslant R_m$, $n = m-1, m, ...,$

for some positive number R_m which is independent of n. Then

$$\|e_{m,n}\|_{\infty} \to 0, \qquad n \to \infty.$$

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